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LESS STRESS MORE SUCCESS

Please note:

- The philosophy of Project Maths is that topics can overlap, so you may encounter Paper 1 material on Paper 2 and vice versa.
- The Exam questions marked by the symbol exam Q in this book are selected from the following:
  1. SEC Exam papers
  2. Sample exam papers
  3. Original and sourced exam-type questions

1 Algebra

aims

- To learn how to factorise and simplify algebraic expressions
- To learn how to change the subject of a formula
- To learn how to use the method of undetermined coefficients
- To learn how to simplify expression which involve surds

Factorising and simplifying expressions

You must be able to factorise expressions using the following methods:

Take out common terms $ab + ad = a(b + d)$	**Factorise by grouping** $ab + ad + cb + cd = (a + c)(b + d)$
Factorise a trinomial $a^2 - 2ab + b^2 = (a - b)(a - b)$	**Difference of two squares** $a^2 - b^2 = (a + b)(a - b)$
Difference of two cubes $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	**Sum of two cubes** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

exam focus

Factorising is a basic and vital skill for you to have throughout your maths course. You must be able to factorise expressions quickly and easily. This will take practice, but it is worthwhile spending time on.

Example

Factorise the following:

(i) $3p^2 + 6pq$ (ii) $6ab + 12bc - 8ac - 9b^2$ (iii) $3x^2 - 12y^2$

Solution

(i) $3p^2 + 6pq$

$3p(p + 2q)$

(ii) $6ab + 12bc - 8ac - 9b^2$

$6ab - 9b^2 - 8ac + 12bc$

$3b(2a - 3b) - 4c(2a - 3b)$

$(2a - 3b)(3b - 4c)$

(iii) $3x^2 - 12y^2$

$3(x^2 - 4y^2)$

$3(x + 2y)(x - 2y)$

Example

Factorise the following:

(i) $2x^2 - 7x - 15$ **(ii)** $64 - 27x^3$ **(iii)** $4a^3 + 32b^3$

Solution

(i) $2x^2 - 7x - 15$

$(2x + 3)(x - 5)$

(ii) $64 - 27x^3$

$(4)^3 - (3x)^3$

$(4 - 3x)(4^2 + (4)(3x) + (3x)^2)$

$(4 - 3x)(16 + 12x + 9x^2)$

(iii) $4a^3 + 32b^3$

$4(a^3 + 8b^3)$

$4((a)^3 + (2b)^3)$

$4(a + 2b)(a^2 - (a)(2b) + (2b)^2)$

$4(a + 2b)(a^2 - 2ab + 4b^2)$

Example

Simplify $\dfrac{x^3 + 7x^2 + 12x}{x^2 + 2x - 3}$.

exam focus

A question asking you to simplify often involves factorising.

Solution

Factorise the top and bottom:

$$\frac{x(x^2 + 7x + 12)}{x^2 + 2x - 3}$$

$$\frac{x(x + 3)(x + 4)}{(x + 3)(x - 1)}$$ (divide top and bottom by $(x + 3)$)

$$\frac{x(x + 4)}{x - 1}$$ (in simplest form)

Example

Simplify $\dfrac{5}{2x - 3} - \dfrac{3}{2x^2 - 3x} - \dfrac{1}{x}$.

Solution

$$\frac{5}{2x - 3} - \frac{3}{x(2x - 3)} - \frac{1}{x}$$ (factorise the denominator)

$$\frac{5(x) - 3(1) - 1(2x - 3)}{x(2x - 3)}$$ (common denominator $x\,(2x - 3)$)

$$\frac{5x - 3 - 2x + 3}{x(2x - 3)}$$

$$\frac{3x}{x(2x - 3)}$$ (divide top and bottom by x)

$$\frac{3}{2x - 3}$$

Example

Simplify $\dfrac{a + b}{\dfrac{1}{a} + \dfrac{1}{b}}$.

Solution

$$\frac{a + b}{\dfrac{1}{a} + \dfrac{1}{b}}$$

$$\frac{a + b}{\dfrac{1}{a} + \dfrac{1}{b}} \times \frac{ab}{ab}$$ (multiply top and bottom by ab)

$$\frac{ab(a + b)}{ab\left(\dfrac{1}{a}\right) + ab\left(\dfrac{1}{b}\right)}$$

$$\frac{ab(a + b)}{b + a}$$ (divide top and bottom by $(a + b)$)

$$ab$$

exam Q

Let $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$, $x \neq \pm 1$, and $g(x) = \dfrac{x^2 + x + 1}{x^2 - x - 2}$, $x \neq -1, 2$.

If $f(x) \div g(x) = ax + b$, find the value of a and b.

Solution

$$f(x) \div g(x) = \frac{x^3 - 1}{x^2 - 1} \div \frac{x^2 + x + 1}{x^2 - x - 2}$$

$$= \frac{x^3 - 1}{x^2 - 1} \times \frac{x^2 - x - 2}{x^2 + x + 1}$$ (invert the second fraction and multiply)

$$= \frac{(x - 1)(x^2 + x + 1)}{(x + 1)(x - 1)} \times \frac{(x - 2)(x + 1)}{x^2 + x + 1}$$ (factorise each part)

$$= \frac{1}{1} \times \frac{(x - 2)}{1}$$ (divide top and bottom by $(x - 1)(x^2 + x + 1)(x + 1)$)

$= x - 2$, which is in the form $ax + b$

Therefore $a = 1$, $b = -2$.

Changing the subject of a formula

When we rearrange a formula so that one of the variables is given in terms of the others, we are said to be **changing the subject of the formula or manipulation of formulae**. The rules in changing the subject of a formula are the same as when solving an equation. That is, we can:

1. **Add** or **subtract** the same quantity to both sides.
2. **Multiply** or **divide** both sides by the same quantity.
3. **Square** both sides, **cube** both sides, etc.
4. Take the **square root** of both sides, take the **cube root** of both sides, etc.

Note: Whatever letter comes after the word 'express' is to be on its own.

exam focus

Changing the subject of a formula (manipulating a formula) is an essential skill, which arises in many sections of the course. These include area and volume, trigonometry and coordinate geometry.

key point

Three common errors made when manipulating formulae are:

1. $\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a + b}$ 2. $\frac{a}{b + c} \neq \frac{a}{b} + \frac{a}{c}$ 3. $a\left(\frac{b}{c}\right) \neq \frac{ab}{ac}$

Example

If $c = \dfrac{b^2 - ac}{b + a}$, express a in terms of the other variables.

Solution

$$c = \frac{b^2 - ac}{b + a}$$

$$(b + a)c = (b + a)\left(\frac{b^2 - ac}{b + a}\right) \quad \text{(multiply both sides by } (b + a))$$

$$(b + a)c = (b^2 - ac)$$

$$bc + ac = b^2 - ac \quad \text{(multiply out brackets)}$$

$$ac = b^2 - ac - bc \quad \text{(subtract } bc \text{ from both sides)}$$

$$ac + ac = b^2 - bc \quad \text{(add } ac \text{ to both sides)}$$

$$2ac = b^2 - bc$$

$$\frac{2ac}{2c} = \frac{b^2 - bc}{2c} \quad \text{(divide both sides by } 2c)$$

$$a = \frac{b^2 - bc}{2c}$$

Example

(i) If $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}$, express v in terms of the other variables.

(ii) Hence, determine the value of v when $f = 15$ and $u = 20$.

Solution

(i)
$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$fuv\left(\frac{1}{f}\right) = fuv\left(\frac{1}{u}\right) + fuv\left(\frac{1}{v}\right) \quad \text{(multiply all parts by } fuv)$$

$$uv = fv + fu$$

$$uv - fv = fv + fu - fv \quad \text{(subtract } fv \text{ from both sides)}$$

$$uv - fv = fu$$

$$v(u - f) = fu \quad \text{(factorise out } v)$$

$$\frac{v(u - f)}{(u - f)} = \frac{fu}{(u - f)} \quad \text{(divide both sides by } (u - f))$$

$$v = \frac{fu}{u - f}$$

(ii) $f = 15$ and $u = 20$: $v = \dfrac{fu}{u - f}$

$$v = \frac{(15)(20)}{20 - 15} = \frac{300}{5} = 60$$

exam Q

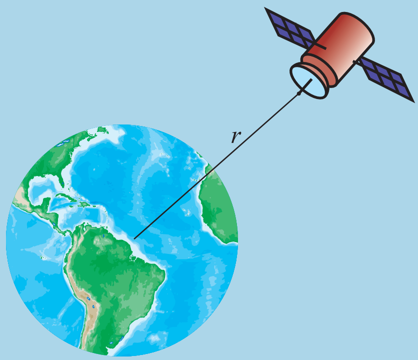
The time taken, in seconds, for a satellite to complete an orbit of the Earth is given by the formula:

$$T = \sqrt{\frac{4\pi^2 r^3}{GM}}$$

where r = radius of rotation from the centre of the Earth

G = universal gravitational constant

M = mass of the Earth.



(i) Express the radius of rotation, r, in terms of the other variables.

(ii) The International Space Station (ISS) orbits the Earth once every 91 minutes. Given that the value for $G = 6{\cdot}67 \times 10^{-11}$ and the mass of the Earth is $6{\cdot}4 \times 10^{24}$, find the radius of rotation of the ISS, correct to the nearest metre.

(iii) Find the height the ISS is above the surface of the Earth, given that the radius of the Earth is 6,371 km.

Solution

(i)

$$T = \sqrt{\frac{4\pi^2 r^3}{GM}}$$

$$T^2 = \left(\sqrt{\frac{4\pi^2 r^3}{GM}}\right)^2 \quad \text{(square both sides)}$$

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

$$GM(T^2) = GM\left(\frac{4\pi^2 r^3}{GM}\right) \quad \text{(multiply both sides by } GM\text{)}$$

$$GMT^2 = 4\pi^2 r^3$$

$$\frac{GMT^2}{4\pi^2} = r^3 \quad \text{(divide both sides by } 4\pi^2\text{)}$$

$$\sqrt[3]{\frac{GMT^2}{4\pi^2}} = r \quad \text{(take cube root of both sides)}$$

(ii) $G = 6{\cdot}67 \times 10^{-11}$, $M = 6{\cdot}4 \times 10^{24}$, $T = 91$ minutes $= 5{,}460$ seconds

key point

Index notation
When dealing with very large or very small numbers, it can be easier to perform calculations if the numbers are expressed in the form $a \times 10^n$, where $0 < a < 10$.

$$\sqrt[3]{\frac{GMT^2}{4\pi^2}} = r$$

$$\sqrt[3]{\frac{(6{\cdot}67 \times 10^{-11})(6{\cdot}4 \times 10^{24})(5{,}460)^2}{4\pi^2}} = r$$

$$\sqrt[3]{3{\cdot}224 \times 10^{20}} = r$$

$$6{,}856{,}625{\cdot}858 = r$$

Thus, the radius of rotation $= 6{,}856{,}626$ m.

(iii) Radius of rotation = radius of Earth + height above Earth

$6{,}856{,}626 = 6{,}371{,}000 +$ height above Earth

$6{,}856{,}626 - 6{,}371{,}000 =$ height above Earth

$485{,}626 =$ height above Earth

Thus, the ISS is at a height of $485{,}626$ m $= 486$ km above the surface of the Earth.

Undetermined coefficients

When two expressions in x (or any other variable) are equal to one another for all values of x, we can equate the coefficients of the same powers of x in the two expressions. This is known as the **principle of undetermined coefficients**.

Method:

1. Remove all fractions and brackets.
2. Form equations by equating coefficients of like terms.
3. Solve the equations to find the coefficients.

key point

It may help to highlight, colour or underline the like terms.
For example:

$$x^2 + 15x + kx - c = x^2 + 3kx + k^3$$

Example

$x^2 - 6x + t = (x + k)^2$, where t and k are constants. Find the value of k and the value of t.

Solution

$x^2 - 6x + t = (x + k)^2$

$x^2 - 6x + t = x^2 + 2xk + k^2$ (multiply out the brackets)

$x^2 - 6x + t = x^2 + 2xk + k^2$ (colour like terms)

Equate like terms:

Terms containing x:	Terms independent of x:
$-6x = 2xk$	$t = k^2$
$-6 = 2k$	$t = (-3)^2$
$-3 = k$	$t = 9$

Thus, $k = -3$, $t = 9$.

Example

The following equation is true for all y.

$$ay^2 + by(y - 4) + c(y - 4) = y^2 + 13y - 20$$

Find the values of the constants a, b and c.

Solution

$ay^2 + by(y - 4) + c(y - 4) = y^2 + 13y - 20$

$ay^2 + by^2 - 4by + cy - 4c = y^2 + 13y - 20$ (multiply out brackets)

$ay^2 + by^2 - 4by + cy - 4c = y^2 + 13y - 20$ (colour like terms)

Equate like terms:

Terms containing y^2:	Terms containing y:	Terms independent of y:
$ay^2 + by^2 = y^2$	$-4by + cy = 13y$	$-4c = -20$
$a + b = 1$ ①	$-4b + c = 13$ ②	$c = 5$ ③

Substitute $c = 5$ into ②:	Substitute $b = -2$ into ①:
$-4b + c = 13$	$a + b = 1$
$-4b + (5) = 13$	$a + (-2) = 1$
$-4b = 8$	$a = 1 + 2$
$b = -2$	$a = 3$

Therefore, $a = 3$, $b = -2$ and $c = 5$.

Surds

Properties of surds:

1. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ **2.** $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ **3.** $\sqrt{a}\sqrt{a} = a$

key point

Simplification of surds

Find the largest possible perfect square number greater than 1 that will divide evenly into the number under the square root. Then use the property $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Example

Express each of the following in its simplest surd form:

(i) $\sqrt{99}$ (ii) $\frac{1}{2}\sqrt{112}$ (iii) $\sqrt{2\frac{1}{4}}$

Solution

(i) $\sqrt{99}$

$\sqrt{9 \times 11}$

$\sqrt{9}\sqrt{11}$

$3\sqrt{11}$

(ii) $\frac{1}{2}\sqrt{112}$

$\frac{1}{2}\sqrt{16 \times 7}$

$\frac{1}{2}\sqrt{16}\sqrt{7}$

$\frac{1}{2}(4)\sqrt{7} = 2\sqrt{7}$

(iii) $\sqrt{2\frac{1}{4}}$

$\sqrt{\frac{9}{4}}$

$\frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2}$

exam focus

The natural display calculators can be very useful when verifying answers to questions on surds.

Example

Express each of the following in its simplest surd form:

$\sqrt{180} + \sqrt{20} - \sqrt{125}$

Solution

$\sqrt{180} + \sqrt{20} - \sqrt{125}$

$\sqrt{36 \times 5} + \sqrt{4 \times 5} - \sqrt{25 \times 5}$

$\sqrt{36}\sqrt{5} + \sqrt{4}\sqrt{5} - \sqrt{25}\sqrt{5}$

$6\sqrt{5} + 2\sqrt{5} - 5\sqrt{5}$

$8\sqrt{5} - 5\sqrt{5}$

$3\sqrt{5}$

key point

Addition and subtraction

Only like surds can be added or subtracted. Express each surd in its simplest form and add or subtract like surds.

Rationalising the denominator

It is poor practice to have surds on the bottom of a fraction. The process of removing a surd from the denominator is called **rationalising the denominator**. To rationalise the denominator, multiply the top and bottom of the fraction by the surd.

If the denominator is a compound surd, such as $a + \sqrt{b}$, you rationalise the denominator by multiplying the top and bottom by the conjugate of the denominator, which is the same as the compound surd, but with one of the signs changed.

$$\frac{x}{a + \sqrt{b}} \times \frac{a - \sqrt{b}}{a - \sqrt{b}}$$

Example

Express $\dfrac{1}{3\sqrt{5}} - \dfrac{1}{2\sqrt{20}}$ in the form $k\sqrt{5}$, where $k \in \mathbb{Q}$.

Solution

$\dfrac{1}{3\sqrt{5}} - \dfrac{1}{2\sqrt{20}}$

$\dfrac{1}{3\sqrt{5}} - \dfrac{1}{2\sqrt{4}\sqrt{5}}$ (break down the $\sqrt{20}$)

$\dfrac{1}{3\sqrt{5}} - \dfrac{1}{2(2)\sqrt{5}}$